Plasma Waves

Mickaël Melzani[*]

March 9, 2014

Contents

1 Introduction

In a plasma, waves can interact with particles, by transferring energy and momentum from the field to the particles and vice-versa. In this way, they are also capable of communicating between the particles themselves: charge separation or currents, originating from particles moved by waves, may induce changes in fields which produce waves. By this, in a complete collisionless way, there is a highly non-linear coupling between particles. For instance, most of the shock waves observed in space are collisionless: without a single collisions but only via particle-wave-particle interactions, directed kinetic energy of the particles is transformed into thermal, isotropic kinetic energy.

Waves are also the fundamental bricks of several plasma phenomena, and thinking of interactions, coupling, turbulence, communication, instabilities, or many other aspects of plasmas in terms of waves, is often enlightening.

2 Waves in a homogeneous plasma

We have previously introduced ideal MHD waves, and seen that they consist in three modes: the fast magnetosonic wave, the slow magnetosonic wave, and the Alfvén wave.

[*]Centre de Recherche Astrophysique de Lyon, École Normale Supérieure de Lyon, France.
mickael.melzani@gmail.com

Their dispersion relation is linear ($\omega/k = \text{cst}$ for a given direction of propagation), and this linearity is reminiscent from the fact that ideal MHD is a low frequency approximation (ω far smaller that the cyclotron and plasma pulsations). Consequently, ideal MHD waves are only valid for low frequencies. At higher frequencies, a new theory must be used.

In this section, we present a short overview of the waves present in a *homogeneous* ion-electron plasma, with a Maxwellian distribution function for particle velocities (non-relativistic), and neglecting the effects of collisions. We give some derivations, but for more the interested reader may refer to the introductory textbooks by Fitzpatrick (2011), Chen, Bellan (2006). We believe our approach useful, because plasma waves studies are often scattered in different chapters depending on the model used, and are seldom presented in a global view as we do here.

In a homogeneous ion-electron plasma, there are two parameters: the strength of the magnetic field, and the temperature. The cold and unmagnetized plasma case is fairly simple, and is presented first. Allowing this plasma to be hot adds some complications, and the venue of Landau damping. Then, the theory used for a cold and magnetized plasma remains fairly simple. The theory for a plasma both hot and magnetized, in the low frequency limit, is MHD. The general case of a hot and magnetized plasma at arbitrary frequencies requires the use of Vlasov equation with a background and perturbed magnetic field and is quite complicated, with many possible waves and damping. It is barely touched on here, and some useful examples may be found in Chen or Fitzpatrick (2011). The complete theory is treated in the reference book of Stix (1992).

Some vocabulary may be useful. Let us consider a Fourier mode $\propto \exp\{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\}$.The equation $\nabla\cdot\boldsymbol{B} = 0$ involves $\boldsymbol{k} \perp \boldsymbol{B}$, so that the magnetic field perturbation vector is always transverse to the wave vector. On another hand, there is no constraint for the electric field perturbation, and we distinguish several cases:

- A **Longitudinal** wave has $\boldsymbol{k} \parallel \boldsymbol{E}$. By definition, an **Electrostatic** wave has no inductive electric field, i.e., $\boldsymbol{E} = -\nabla\Phi$ or $\nabla\wedge\boldsymbol{E} = 0$[1]. This last relation involves $\boldsymbol{k}\wedge\boldsymbol{E} = 0$, so that longitudinal and electrostatic are equivalent. Since $\partial_t\boldsymbol{B} = \nabla\wedge\boldsymbol{E} = 0$, there is *no magnetic field perturbation.*
- A **Transverse** wave has $\boldsymbol{k} \perp \boldsymbol{E}$. It involves that $\nabla\cdot\boldsymbol{E} = 0$, so that there is *no density perturbation.* **Inductive** means that the electromagnetic field is created only by currents (and not by charges). Thus, $\nabla\cdot\boldsymbol{E} = 0$. This last point shows that transverse is equivalent to inductive.

A notion, somehow opposed to electrostatic, that of **electromagnetic** waves. It means that there is both an electric and magnetic field perturbation. Since $\nabla\cdot\boldsymbol{B} = 0$, the perturbation in the magnetic field is always transverse, but there is no constraints for the electric field perturbation, and it can be either longitudinal, transverse, or both (as in the inertial Alfvén wave or in the X-wave).

Special points in the dispersion relation can be mentionned.

- **Resonance**: There is a resonnance when $n = ck/\omega \to \infty$, or $\omega/k \to 0$. The wave is usually absorbed.
- **Cutoff**: There is a cutoff when $n = ck/\omega \to 0$, or $\omega/k \to \infty$. The wave is usually reflected.

[1] Forgetting about waves for a moment, we recall that an electrostatic field is a field $\boldsymbol{E}$ that has no temporal dependence. It is then showed, from Maxwell equation, that $\nabla\wedge\boldsymbol{E} = 0$ and that it can be derived from a scalar potential. There could however be time dependent electric field that obeys $\nabla\wedge\boldsymbol{E} = 0$. This is the case of our wave, and it is why it is called an “electrostatic” wave, even though it is time dependent. For properties of electrostatic waves, and in particular energy considerations, see McDonald (2002).

It may puzzle the reader that most of the derivations presented here make specific assumptions. For example in a unmagnetized and hot plasma, we assume longitudinal waves ($\boldsymbol{E} \wedge \boldsymbol{k} = 0$) to find Langmuir waves, and transverse waves ($\boldsymbol{E} \cdot \boldsymbol{k} = 0$) to find electromagnetic waves. How to be sure that these waves are not coupled, and that there is no waves that are both transverse and longitudinal? To answer this important question, one should make derivations without a priori assumptions. The general scheme is usually as follows: one write down the equations used (1st hypothesis: the model), linearize around an equilibrium (2nd hypothesis: the equilibrium, often homogeneous), swap to Fourier space (with an arbitrary wavevector direction $\boldsymbol{k}$), manipulate equations to end with a matrix equation of the kind $M\boldsymbol{q} = 0$, with $\boldsymbol{q}$ a vector containing relevant variables ($\boldsymbol{v}$, $\boldsymbol{B}$, ...). Non-trivial solutions are obtained if and only if $\det M = 0$, and this gives the dispersion relation. The different plasma waves, or modes, are found by finding, for each value of $\omega_i(\boldsymbol{k})$ that cancels $\det M$, the associated vector satisfaying $M\boldsymbol{q_i} = 0$. Each q_i, with its dispersion relation $\omega_i(\boldsymbol{k})$, will be an independant mode, and all modes are thus found[2].

2.1 Cold and unmagnetized plasma

Overview

This is the simplest case. The only allowed modes are the electrostatic Langmuir oscillation and the electromagnetic wave.

- The electrostatic Langmuir oscillation is an oscillation (not a wave, it has a zero group velocity and does not propagate) at the plasma pulsation. It is electrostatic, and the oscillation is longitudinal to the wavevector. The pulsation of oscillation is the plasma pulsation, given by

$$\omega_P^2 = \sum_s \omega_{ps}^2. \tag{1}$$

  The sum extends over all plasma species s, and the ω_{ps} are the individual plasma pulsations ($\omega_{ps} = \sqrt{n_s q_s^2/(\epsilon_0 m_s)}$). In the case of an ion-electron plasma, $\omega_P = \omega_{pe}(1 + m_e/m_i)^{1/2} \simeq \omega_{pe}$.

- The electromagnetic wave is transverse (oscillation of the fields perpendicular to the propagation direction), and is a standard vacuum electromagnetic wave, but modified by the response of the plasma[3]. It is incompressible ($n_{1s} = 0$ and $\nabla \cdot \boldsymbol{v}_{1s} = 0$). Its dispersion relation is

$$\omega^2 = k^2c^2 + \omega_P^2. \tag{2}$$

  At high frequency, it becomes the electromagnetic wave of vacuum ($\omega \simeq kc$) because the plasma has no time to respond. At frequencies below the plasma pulsation, it is absorbed by the plasma and cannot propagate[4].

The dispersion relations of these waves are shown on the upper-right panel of Fig. 2.

[2] A different approach, develloped in the previous Chapter on MHD waves, consists in writing the system of equations in a flux-conserving way, when it is possible. The wave modes are then the eigenvectors of the Jacobian of the flux matrix, and the associated eigenvalues are the wave speeds.

[3] Such waves propagate in the first layers of the ionosphere, and it is the ionosphere that allows the transmission of radio waves over large distances on Earth. It is because radio waves are so efficiently transfered that the very existence of the ionosphere was first inferred, and its properties investigated by the launching of such waves (Appleton, 1948).

[4] This property allows to measure plasma number densities: the frequency is risen from low values, and when the wave is transmitted, it is that $\omega = \omega_{pe}$, hence n_e.

Some derivations and more details

The oscillations at ω_{pe} of electrons around the electrostatic field created by the ions in a cold plasma can be derived from the two-fluid equations by assuming a longitudinal exitation. This will be done in Sect. 2.2. Here we present a simple derivation from the equation of motion of an electron. More specifically, if a slab of electron of surface S is slightly displaced of a length δx from the neutral equilibrium, then the ions where the slab was at the begining will create a restoring electric field. The ions are in number $nS\delta x$, creating a surface charge density $en\delta x$ and thus an electric field $en\delta x/\epsilon_0$. The equation of motion applyed to a single electron moving with the slab then gives $m_e d^2\delta x/dt^2 = -e(en\delta x/\epsilon_0)$, and we have the result. The expression of the full plasma pulsation stated in equation 1 is obtained by allowing the ions to move. Note that in deriving the electric field, we have assumed that the slab is infinite in parallel extension, so that $\boldsymbol{E}$ is perpendicular to the slab. If the plasma is of finite extension, then this will not be the case, there will be fringing electric fields and the oscillations will propagate (Chen).

The electromagnetic wave can be obtained with the two-fluid model, with $s = i$ or e:

$$\partial_t n_s + n_s \nabla \cdot \boldsymbol{u}_s = 0, \tag{3a}$$

$$m_s n_s \left(\partial_t + \boldsymbol{v} \cdot \nabla\right) \boldsymbol{v} = q_s n_s (\boldsymbol{E} + \boldsymbol{v} \wedge \boldsymbol{B}) - \nabla P_s. \tag{3b}$$

The equilibrium (subscript 0) is taken so that for both species, T_0, $n_0 = \text{cst}$, $\boldsymbol{E}_0 = \boldsymbol{B}_0 = \boldsymbol{v}_0 = 0$. We linearize these equations around equilibrium: $n = n_0 + n_1$, $P = P_0 + P_1$, $\boldsymbol{v} = \boldsymbol{v}_1$, $\boldsymbol{E} = \boldsymbol{E}_1$, $\boldsymbol{B} = \boldsymbol{B}_1$. We explicitly search for transverse waves, i.e., waves for which $\boldsymbol{k} \cdot \boldsymbol{B}_1 = \boldsymbol{k} \cdot \boldsymbol{E}_1 = 0$. Since $\nabla \cdot \boldsymbol{B}_1 = 0$, the first constraint is always satisfied. The second implies $q_i n_{1i} + q_e n_{1e} = \epsilon_0 \nabla \cdot \boldsymbol{E} = 0$, so that there is no charge separation. Since ions and electrons evolve on different time scales, it actually implies no density variation at all: $n_{1i} = n_{1e} = 0$. The continuity equation then leads to $\nabla \cdot \boldsymbol{v}_{1s} = 0$, and the wave is incompressible. The linearized momentum equation is:

$$m_s n_{s0} \partial_t \boldsymbol{v}_{s1} = q_s n_{s0} \boldsymbol{E}_1 - \nabla P_{s1}. \tag{4}$$

The divergence part of this equation is not interesting because of incompressibility. The rotational part leads to

$$m_s n_{s0} \partial_t \nabla \wedge \boldsymbol{v}_{s1} = q_s n_{s0} \nabla \wedge \boldsymbol{E}_1 = -q_s n_{s0} \partial_t \boldsymbol{B}_1, \tag{5}$$

and we note that the pressure disappears. Integrating, multiplying by q_s, it yields $m_s q_s n_{s0} \nabla \wedge \boldsymbol{v}_{s1} = -q_s^2 n_{s0} \boldsymbol{B}_1$, and summing over species, using $\boldsymbol{J}_1 = \sum_s n_{0s} q_s \boldsymbol{v}_{1s}$ and the definition of the plasma pulsation (Eq. 1), we have:

$$\nabla \wedge \boldsymbol{J}_1 = -\epsilon_0 \sum_s \frac{q_s^2 n_{0s}}{\epsilon_0 m_s} \boldsymbol{B}_1 = -\epsilon_0 \sum_s \omega_{ps}^2 \boldsymbol{B}_1 = -\epsilon_0 \omega_P^2 \boldsymbol{B}_1 \tag{6}$$

We now use Maxwell-Ampère equation, $\boldsymbol{J}_1 = \mu_0^{-1} \nabla \wedge \boldsymbol{B}_1 - \epsilon_0 \partial_t \boldsymbol{E}_1$, so that:

$$\nabla \wedge (\nabla \wedge \boldsymbol{B}_1) - \epsilon_0 \partial_t \nabla \wedge \boldsymbol{E}_1 = -\epsilon_0 \omega_P^2 \boldsymbol{B}_1. \tag{7}$$

The last step is to replace $\nabla \wedge \boldsymbol{E}_1$ by $-\partial_t \boldsymbol{B}_1$, to end with

$$\nabla^2 \boldsymbol{B}_1 - \frac{1}{c^2} \partial_{tt}^2 \boldsymbol{B}_1 - \frac{\omega_P^2}{c^2} \boldsymbol{B}_1 = 0. \tag{8}$$

With Fourier modes, this immediately leads to the dispersion relation $\omega^2 = k^2 c^2 + \omega_P^2$.

2.2 Hot and unmagnetized plasma

Overview

From the previous case (cold and unmagnetized), one obtains more modes if the temperature is made finite. The dispersion relations of the modes in this case are shown in the lower-right panel of Fig. 2.

- The Debye length, that was zero at zero temperature, now becomes finite. Physically, a finite Debye screening comes from thermal fluctuations: at zero temperature the screening is perfect because particles can adjust so as to cancel the potential everywhere, but for a finite temperature, thermal motions forbid this perfect screening, and potentials of the order of T/e can leak out of the screening cloud over a distance $\sim \lambda_D$, resulting in a finite Debye lentgh below which the plasma is not neutral. This is indeed seen in the expression $\lambda_{Ds} = \sqrt{T_s/m_s}/\omega_{ps} = \sqrt{\epsilon_0 T_s/(n_s q_s^2)}$.
- The electromagnetic wave is not modified.
- However, the electrostatic Langmuir oscillation now becomes a wave (non zero group velocity) due to thermal effects. Its dispersion relation is given by (with $v_{th,e} = \sqrt{T_e/m_e}$):
$$\omega^2 = \omega_{pe}^2 + 3\boldsymbol{k}^2 v_{th,e}^2. \tag{9}$$
- A new mode – the ion sound wave – appears. This mode is a low frequency mode, electrostatic, longitudinal, and compressible. It propagates because of two effects:
  - Thermal motions of the ions can spread the wave, just as for a classical sound wave, at speed $\sqrt{\gamma_i T_i/m_i}$, with γ_i the adiabatic index of the ions[5].
  - The ions communicate via electric fields. Ion packets are positively charged, but the electrons screen this charge excess and let escape only a potential a fraction of T_e (this is Debye screening of charge excess). The ions are called back by this potential, and overshoot because of their inertia. The result is a propagation at velocity $\sqrt{T_e/m_i}$, because electrons controls the repelling force and ions the inertia.

  With these two effects, the total dispersion relation is:
$$\begin{aligned} \frac{\omega}{k} &= \left(\frac{T_e}{m_i} \frac{1}{1+k^2\lambda_D^2} + \frac{\gamma_i T_i}{m_i} \right)^{1/2} \\ &\underset{\lambda \gg \lambda_D}{\simeq} \left(\frac{T_e + \gamma_i T_i}{m_i} \right)^{1/2} \equiv c_s \\ &\underset{\lambda \ll \lambda_D}{\rightarrow} \frac{\omega_{pi}}{k} \quad (\text{if } T_i = 0), \end{aligned} \tag{10}$$
  with $\lambda = 1/k$. If the electron velocity is $T_e = 0$, the screening is perfect and ions cannot communicate via electric field. If $T_i = 0$, the thermal spread cannot drive the wave. Consequently if both temperatures are null, the wave does not exist.

Landau damping

When the temperature is rose from zero, a new phenomena appears: Landau damping. It consists in the fact that particles travelling at a velocity close to the phase speed of a

[5] Note however that no collisions are involved here, it is just a spread via kinetic pressure.

wave can interact with the wave and exchange energy with it, just like a surfer on a wave is carried by the flow. This is not possible in a cold plasma, because particles have no velocity at all (except for the very particles moving with the wave), and such particles just oscillate in the fields with no net energy gain. However, in a hot plasma there are particles at all speeds (following the velocity distribution function $f(\boldsymbol{v})$). A simple picture is then to consider particles with velocity infinitely close to the phase speed, $v = (\omega/k)^+$ or $(\omega/k)^-$, and to show that these particles tend to align their velocity with the wave speed: particles slightly above the phase speed will be decelerated by the wave and thus give energy to the wave, while particles having a velocity slightly below will be accelerated and take energy to the wave. Decay or growth of the energy of the wave thus depends on whether there are more particles just above the phase speed than just below, or vice versa, i.e., depends on the sign of $f'(v = \omega/k)$. For a Maxwellian distribution function, there are always more slow particles than fast ones, and the wave is absorbed by the plasma. If the distribution function has a negative slope somewhere, for example because there is a hot beam of particles, energy can be given to the waves and lead to an instability. This will be discussed in the Chapter on instabilities.

Of course, all waves are not subject to Landau damping. There should be enough particles close to the phase speed, which means that if the phase speed is larger than the thermal velocity, the damping is weak (simply because for $v \gg v_{th}$ in a Maxwellian distribution, there is an exponentially small number of particles contributing to the energy exchange).

In our case, Langmuir waves exist only for phase speeds at least several times $v_{th,e}$ and are Landau damped at lower phase speeds (with a rate given below in equation 23). Ion sound waves are also Landau damped because they are low-frequency, except for $T_e \gg T_i$ or $Z_i \gg 1$. Note that waves with a phase velocity higher than c cannot be Landau damped, because there are no particles matching their phase speed. This is the case of the electromagnetic waves.

Landau damping is discussed in Sec. **??**, by considering particles with $v = (\omega/k)^+$ or $(\omega/k)^-$, which is also the approach of Bellan (2006). Fitzpatrick (2011) and with more details Chen consider another viewpoint, and include in their physical explanation of Landau damping particles close to the phase speed, but not infinitely so. Particles with a velocity above the wavespeed actually both gain and loose energy, but those which lose energy have their velocity coming closer to the wavespeed, thus interacting more, while those which gain energy have their velocity going further away from the wavespeed, thus interacting less with the wave: on average, particles with a larger velocity than the wavespeed lose energy. The exact opposite holds for particles having a smaller velocity: on average, they gain energy at the expense of the wave. The two viewpoints are thus the same[6].

The Landau damping discussed up to now, presented in Sec. **??**, and for which the rate of equation 23 holds, is *linear* Landau damping. It is derived on the basis of linearized equations, and involve particles freely sreaming in straight lines. It is valid for small initial perturbation amplitudes. At higher wave amplitudes, a significant fraction of particles can

[6]Technically, the differences between the two approaches are subtle and may require some explanations. First, Chen shows that a particle travelling in a sinusoidal potential field does not, on average, gain energy. This does not hold in the approach of Bellan and of Sec. **??**, because they consider particles with $v = (\omega/k)^+$ or $(\omega/k)^-$ travelling at the wavespeed, for which a temporal average makes no sense. Second, Chen insists on the fact that the problem heavily depends on the initial conditions, and that particles with a non-zero averaged energy gain are those that, at time t, have moved from less than $\sim$ half a wavelength. The weight of the contributing particles is then his $\sin u/u$ function with $u = \pi/t$. The link between Chen and Bellan is made when the latter takes the limit $\sin u/u = \delta(u)$ for $u \to +\infty$, i.e., $t \to 0$, i.e., particles contributing only for $v = (\omega/k)^+$ or $(\omega/k)^-$. In the phase space of figure 7-24 of Chen, Bellan considers only the particles very near the X-point.

be trapped in the wells of the wave, and bounce back and forth in it while being carried by the wave. These particles will exchange energy with the wave, and lead (depending on the conditions) to a damping of the wave. This is *non-linear* Landau damping.

Some derivations and more details

We give the demonstration for the electrostatic waves based on two-fluid equations[7]. Insights can be gained concerning the relation between sound waves, Langmuir waves, and Debye screening, which are all electrostatic modes. The following derivation is inspired from Bellan (2006).

Using the two-fluid model, we will soon encounter problems with the pressure: there is no way with a fluid model and its equations of conservation of particle and momentum to link a density perturbation to the temperature disturbance. One should either use the energy equation with assumptions on the heat transfer to close the system, or use a direct hypothesis about the plasma transformation, namely, that it is adiabatic or isothermal. For an adiabatic transformation, one has $P/n^\gamma = \mathrm{cst}$, so that perturbations in pressure and density are linked by $dP/P = \gamma dn/n$, i.e., $dP = \gamma T dn$ (we used $P = nT$). For an isothermal transformation, it is evident that $dP = T dn$. We thus generally use $dP = \gamma T dn$, with:

- For an isothermal process, $\gamma = 1$. Physically, a process is isothermal if the particles move fast enough to smooth any temperature gradient. If the typicall velocity of the process is v, it translates into $v \ll v_{th,s}$ with $v_{th,s}$ the thermal velocity of ions or of electrons, depending on which species behaves isothermally.
- For an adiabatic process, $\gamma = (N+2)/N$, with N the number of degrees of freedom of the plasma species s. For particles with no internal degree of freedom, $\gamma = 5/3$ in three dimensions, 2 in 2D, 3 in 1D. Physically, a process is adiabatic if it evolves on a timescale so fast that particles do not have time to transfert heat to smooth gradients. If the typicall velocity of the process is v, it translates into $v \gg v_{th,s}$ with $v_{th,s}$ the thermal velocity of ions or of electrons, depending on which species behaves adiabatically.

We are now ready to use the equations for the plasma. We will use the conservation of particle number and of momentum, for each species s, and Poisson's equation:

$$\partial_t n_s + n_s \nabla \cdot \boldsymbol{u}_s = 0, \tag{11a}$$

$$m_s n_s \left(\partial_t + \boldsymbol{v} \cdot \nabla\right) \boldsymbol{v} = q_s n_s (\boldsymbol{E} + \boldsymbol{v} \wedge \boldsymbol{B}) - \nabla P_s, \tag{11b}$$

$$\nabla \cdot \boldsymbol{E} = (n_e q_e + n_i q_i)/\epsilon_0. \tag{11c}$$

The equilibrium (subscript 0) is taken so that for both species, T_0, $n_0 = \mathrm{cst}$, $\boldsymbol{E}_0 = \boldsymbol{B}_0 = \boldsymbol{v}_0 = 0$. We linearize these equations around equilibrium: $n = n_0 + n_1$, $P = P_0 + P_1$, $\boldsymbol{v} = \boldsymbol{v}_1$, $\boldsymbol{E} = \boldsymbol{E}_1$, and we assume no magnetic perturbation because we specifically search for electrostatic waves.

We also consider a perturbation at a single frequency, because any solution of the linearized equations can be built as a sum of such components: $\boldsymbol{E}_1 = \tilde{\boldsymbol{E}}_1 \exp\{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\}$, and similarly for all other perturbations (subscript 1). As excpected of electrostatic waves, $\nabla \wedge \boldsymbol{E}_1 = -\partial_t \boldsymbol{B}_1 = 0$, so that $\boldsymbol{k} \wedge \boldsymbol{E}_1 = 0$ and the wavevector is longitudinal. We thus have $\boldsymbol{k} \cdot \boldsymbol{E}_1 = k E_1$.

[7]Vlasov-Maxwell system leads to the same dispersion relations, but would in addition include Landau damping. Also, a simpler derivation than presented here (but less precise concerning the dispersion relations) can be found in Chen.

The linearized momentum equation is:

$$m_s n_{s0} \partial_t \boldsymbol{v}_{s1} = q_s n_{s0} \boldsymbol{E}_1 - \nabla P_{s1}. \tag{12}$$

The rotational part of this equation simply gives $\nabla \wedge \boldsymbol{v}_{s1} = 0$. The divergence leads to

$$m_s n_{s0} \partial_t \nabla \cdot \boldsymbol{v}_{s1} = q_s n_{s0} \nabla \cdot \boldsymbol{E}_1 - \nabla^2 P_{s1}. \tag{13}$$

We argued that the pressure perturbation is linked to the density perturbation via $dP = \gamma T dn$, so that here $P_{s1} = \gamma_s T_{s0} n_{s1}$. Using the conservation of particle number, $\partial_t n_{s1} + n_{s0} \nabla \cdot \boldsymbol{v}_{s1} = 0$, in order to replace the velocity divergence, we have:

$$m_s \partial_{tt}^2 n_{s1} = -q_s n_{s0} \nabla \cdot \boldsymbol{E}_1 + \gamma_s T_{s0} \nabla^2 n_{s1}. \tag{14}$$

Swapping for Fourier modes, it yields:

$$m_s \omega^2 n_{s1} = q_s n_{s0} ikE_1 + \gamma_s T_{s0} k^2 n_{s1}. \tag{15}$$

We thus obtain the density perturbation:

$$n_{s1} = \frac{q_s n_{s0}}{m_s} ikE_1 \times \frac{1}{\omega^2 - \gamma_s T_{s0} k^2 / m_s}. \tag{16}$$

The next and last step is to use Poisson equation, where we insert the above result for the densities:

$$ikE_1 = \frac{1}{\epsilon_0} \sum_s \frac{q_s^2 n_{s0}}{m_s} ikE_1 \times \frac{1}{\omega^2 - \gamma_s T_{s0} k^2 / m_s}. \tag{17}$$

All in all, the dispersion relation is:

$$1 + \sum_s \frac{-\omega_{ps}^2}{\omega^2 - \gamma_s T_{s0} k^2 / m_s} = 0. \tag{18}$$

with the plasma pulsations $\omega_{ps}^2 = q_s^2 n_{s0} / (m_s \epsilon_0)$. It can also be written $1 + \chi_i + \chi_e = 0$, with χ_s the susceptibility of each species defined in the above sum. The limits of the χ_s are as follow:

- Adiabatic limit, i.e., $\omega/k \gg v_{th,s}$ (with $v_{th,s} = \sqrt{T_{0s}/m_s}$): then $\gamma = 3$ because the compression/dilatation occurs only along the electric field, which is along the wave propagation, so that $N = 1$. Then:

$$\chi_s \simeq \frac{\omega_{ps}^2}{v_{th,s}^2 k^2} = \frac{1}{\lambda_{Ds}^2 k^2}, \tag{19}$$

  with λ_{Ds} the Debye length.

- Isothermal limit, i.e., $\omega/k \ll v_{th,s}$ (with $v_{th,s} = \sqrt{T_{0s}/m_s}$): then $\gamma = 1$ and

$$\chi_s \simeq -\frac{\omega_{ps}^2}{\omega^2} \left(1 + 3 v_{th,s}^2 \frac{k^2}{\omega^2} \right). \tag{20}$$

For our waves, three cases are possibles, depending on the ordering between ω/k and the ion and electron thermal velocities.

- Ions and electrons isothermal: $\omega/k \ll v_{th,e},\ v_{th,i}$.
  The dispersion relation then reads

$$k^2 = 1 + \sum_s 1/\lambda_{Ds}^2. \tag{21}$$

  This is **Debye screening**. The dispersion relation has no frequency dependance, and has for solution $k = \pm i/\lambda_D$, resulting in a spatial damping $\propto \exp(-x/\lambda_D)$ of all frequencies $\omega \ll kv_t = v_{th,s}/\lambda_D = \omega_p$. It shows that when $\omega/k \ll v_{th,e},\ v_{th,i}$, the plasma reaches the steaty-state limit where the perturbations are screened. It also shows that a species can contribute to screening of a perturbation of phase velocity v_φ only if its thermal velocity is higher. For example, since usually $v_{th,e} \gg v_{th,i}$, ions cannot screen electron induced perturbations.

- Ions and electrons adiabatic: $\omega/k \gg v_{th,e},\ v_{th,i}$.
  The dispersion relation then reads

$$\omega \simeq \omega_{pe}\left(1 + \frac{3}{2}\boldsymbol{k}^2\lambda_{D_e}^2\right). \tag{22}$$

  This is the **Langmuir waves**.

  Some more remarks on these waves may be of interest. At zero temperature, they are the oscillations, at the plasma frequency, of the electrons in the electrostatic field created by the ions: the ions cannot follow the quick electron displacements and there is a restoring electric field; then because of their inertia the electrons overshoot, and there is an oscillation[8]. The wave is then non-propagating ($d\omega/dk = 0$). When the electronic temperature is finite, the pressure acts against these oscillations, and the frequency is slightly corrected with a term that depends on the wavelength: the group velocity is then small but non-zero. The physical reason is that electrons have a thermal velocity spread and can propagate information from one layer to another. At large k (small λ), information propagates at $\sqrt{3/2}v_{te}$, roughly the thermal velocity. At small k (large λ), the group velocity is much smaller, because the density gradients are very small and the thermal motions propagate information very slowly.

  Longitudinal waves cannot exist in vacuum, and are allowed only by the dielectric response of the plasma. They still exist if the plasma is not Maxwellian, but with a different dispersion relation. However, purely longitudinal waves do not exist if the distribution is not isotropic.

  We note that the assumption of adiabaticity implies, because $\omega \sim \omega_{pe}$, that $\lambda \gg \lambda_{De}$: the wavelength is far larger than the Debye length. On another hand, the motion of a typicall particle moving at the thermal velocity during one wave period is $l = v_{th,s}/\omega_{ps} = \lambda_{Ds}$. Particles thus move far less than one wavelength during one wave period. This is a necessary condition for adiabaticity.

  Finally, Langmuir waves are Landau damped. Vlasov theory gives a Landau damping rate (Lifshitz and Pitaevskii (1981, equ.32.7), Bellan (2006, equ.5.89))

$$\begin{aligned}\gamma &= \sqrt{\frac{\pi}{8}}\frac{\omega_{pe}}{(k\lambda_{De})^3}\exp\left(-\frac{\omega^2}{k^2 v_{th,e}^2}\right)\\ &= \sqrt{\frac{\pi}{8}}\frac{\omega_{pe}}{(k\lambda_{De})^3}\exp\left(-\frac{1}{2(k\lambda_{De})^2} - \frac{3}{2}\right)\end{aligned} \tag{23}$$

  They are damped by the electrons of velocity around the phase velocity, i.e., $\boldsymbol{k}\cdot\boldsymbol{v} = kv_\varphi = \omega \sim \omega_{pe}$. Consequently, the damping is efficient only when a lot of electrons

[8] Note that electron finite mass and non-neutrality are essential here.

can contribute, that is, when $v_{th,e} \gg \omega/k \sim \omega_{pe}/k$. This is equivalent to $k\lambda_{De} \gg 1$: when the wavelength is smaller than the Debye length, the wave is damped. This can actually be seen on the damping rate: when $k\lambda_{De} \ll 1$ (wavelength large), the damping rate is exponentially small compared to the frequency and the wave actually exists. When $k\lambda_{De} \sim 1$, the damping rate is of the order of the frequency and the wave is not defined anymore. Note that given the functional form of the Maxwellian distribution, there is an exponentially smaller number of electrons around v_φ when the temperature dwindles. Ultimately, there is no damping when $T_e = 0$.

- Ions adiabatic, electrons isothermal: $v_{th,i} \ll \omega/k \ll v_{th,e}$. The dispersion relation reads:

$$\frac{\omega}{k} = \left(\frac{T_e}{m_i} \frac{1}{1 + k^2\lambda_D^2} + \frac{\gamma_i T_i}{m_i} \right)^{1/2}. \tag{24}$$

  This is **ion sound waves**. Some remarks may be useful.

  At low frequency or large wavelength (small k), the waves present no dispersion. At small wavelength (large k), the electrons are not able to correctly screen the ion induced electric field, and the wave turns into an ion oscillation (no propagation) at the ion plasma pulsation (if $T_i = 0$).

  Note that if $T_e = 0$, the screening is perfect and no electric field is produced by ion bunching. The propagation is then only due to ion thermal motion. Conversely, if $T_i = 0$, there can still be a propagation because of the electric field created by the bunching (if T_e is finite).

  Ion sound waves are fundamentally different from electron plasma waves: ion sound waves are low-frequency waves with constant phase velocity, and exist only because of thermal fluctuations; whereas electron plasma waves are zero temperature high-frequency non-propagating waves, and propagates only because of a small correction due to thermal motions.

  Finally, these waves are strongly Landau-damped (actually, non-existant) if $T_i > T_e$. In order to exist, one should have at least $T_i < T_e/5$. This is because for $T_i \sim T_e$, the phase speed is roughly $\sqrt{T_i/m_i}$, which is the thermal ion speed, so that many ions can contribute to the damping. But when $T_i \ll T_e$, the phase speed is roughly $\sqrt{T_e/m_i}$: ions cannot contribute to the damping because then $v_{th,i} \ll \sqrt{T_e/m_i}$.

2.3 Cold and magnetized plasma

If we come back to the first case (cold and unmagnetized: $T = 0$, $B_0 = 0$), and let the background magnetic field be finite, we obtain a more complex situation. First, it is anisotropic, so that the modes allowed depend on the direction of propagation relative to $\boldsymbol{B}_0$. Several new characteristic pulsations appear, and are a mix between the plasma pulsations and the cyclotron pulsations. They are defined in section 2.6. The two simplest cases are for parallel and perpendicular propagations (the two upper-left diagrams of Fig. 2).

Parallel propagation

- The electrostatic Langmuir oscillation is longitudinal, so that the oscillations are parallel to the background magnetic field. It is thus unchanged.
- The electromagnetic wave, which was of arbitrary transverse polarization, is split in two branches: one which is circularly left polarized and starts from ω_L, the other

circularly right polarized and starting from ω_R[9].

- Two new branches appear: one from $\omega = 0$ to ω_{ci} is left polarized and absorbed by ion cyclotronic motion as $\omega \to \omega_{ci}$; the other from $\omega = 0$ to ω_{ce} is right polarized and absorbed by electron cyclotronic motion as $\omega \to \omega_{ce}$.

  These absorptions occur when the frequency of the circular wave matches the gyrofrequencies of either the ions or the electrons.

The four branches now present (see Fig. 2) are grouped two by two according to whether they are circularly right or left polarized. They are called L and R modes. The intermediate frequency part of the R-wave is called a whistler wave[10].

Perpendicular propagation

- The electromagnetic wave is transverse, so that when it propagates perpendicularly to $\boldsymbol{B}_0$, it involves oscillations parallel to $\boldsymbol{B}_0$. It is thus unchanged. It is now called the O-mode, or ordinary mode.
- However, the electrostatic Langmuir oscillation is modified and results in new branches, grouped under the name X-wave (or extraordinary wave). They are drawn in orange on the figure.

  For $\omega \to \omega_{uh}$, there is a resonnance and the wave is absorbed. It is actually converted into the electrostatic upper-hybrid oscillation of the electrons perpendicular to $\boldsymbol{B}_0$. The same occurs for $\omega \to \omega_{lh}$, where the wave is converted into the electrostatic lower hybrid ion oscillation (see Chen).

Oblique propagation

When the wavevector makes an arbitrary angle with $\boldsymbol{B}_0$, the X, O, L and R waves become coupled. There are however always at most two modes allowed, for example a X-R-coupled mode and a O-L-coupled mode, or a X-L-coupled mode and a O-R-coupled mode. Each mode can have several branches (for example the X-mode has 3 branches, the O-mode one, etc). The exact situation depends on the respective values of ω, ω_{pe} and ω_{ce}, and is summed up in the famous CMA diagram. See for example Bellan (2006) or Chen.

2.4 Hot and magnetized plasma in the low frequency limit

Overview

By low frequency, we mean $\omega \ll \omega_{ci}$. The theory used is ideal MHD, and we have already described these waves. Their dispersion relation is linear ($\omega/k = \text{cst}$) and only depends on the orientation of the wavevector with respect to the magnetic field. This is summed up in the the two lower-left diagrams of Fig. 2.

These modes are:

[9] The fact that these branches have different phase velocities is at the origin of Faraday rotation: when traveling through the plasma, the polarization of a linearly polarized wave turns from an angle. The rotation angle is proportional to the magnetic field, and allows in some cases to measure its strength.

[10] Whistler waves were first observed as disturbances during radio transmissions, from where they get their name. They are produced by disturbances in the upper ionosphere (lightning), and propagate parallel to the magnetic field of the Earth in the magnetosphere, up to the ground in the polar regions. They are a source of information on the state of the magnetosphere and are analyzed by some polar observatories. They are also of importance in various domains, such as magnetic reconnection or turbulent transport in accretion disks (via the magnetorotational instability).

- The fast magnetosonic wave, also named fast wave or compressional Alfvén wave.

  As $T \to 0$, its phase speed is equal to the Alfvén velocity and one recovers for perpendicular propagation the low-frequency part of the low-frequency branch of the X-wave (see the black arrows on the figure), and for parallel propagation the low-frequency part of the low-frequency branch of the R and L waves.

  As $B_0 \to 0$, its phase speed is equal to the sound speed and one recovers the low frequency part of the ion sound wave.

- The Alfvén wave, also named shear Alfvén wave or intermediate wave. This wave is transverse ($\boldsymbol{V}_1 \cdot \boldsymbol{k} = 0$), not compressible ($\rho_1 = P_1 = 0$), and $\boldsymbol{B}_1 \propto \boldsymbol{V}_1$. It is not perturbed by temperature effects, so that it does not change when $T \to 0$. It is absent for perpendicular propagation, and for parallel propagation it becomes degenerate with the fast wave. When $B_0 \to 0$, it disappears.

- The slow magnetosonic wave, also named slow wave. If $T \to 0$ or $B_0 \to 0$, it disappears.

Some derivations and more details

In order to illustrate what we meant in the introduction by finding all the modes allowed by the model at once, we derive the different MHD waves. We mostly follow the work of Fitzpatrick (2011). The set of model equations is:

$$\partial_t \rho + \nabla \cdot \rho \boldsymbol{V} = 0 \tag{25a}$$
$$\rho \partial_t \boldsymbol{V} + \boldsymbol{V} \cdot \nabla \boldsymbol{V} = -\nabla P + \mu_0^{-1} (\nabla \wedge \boldsymbol{B}) \wedge \boldsymbol{B} \tag{25b}$$
$$-\partial_t \boldsymbol{B} + \nabla \wedge (\boldsymbol{V} \wedge \boldsymbol{B}) = 0 \tag{25c}$$
$$P/\rho^\gamma = \text{cst}. \tag{25d}$$

We linearize around an equilibrium $\boldsymbol{B}_0$, ρ_0, $P_0 = \text{cst}$, $\boldsymbol{V}_0 = 0$, and introduce perturbed quantities with a subscript 1, described by Fourier modes $\propto \exp\{i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)\}$. The linearized equations are then

$$\partial_t \rho_1 + \rho_0 \nabla \cdot \boldsymbol{V}_1 = 0 \tag{26a}$$
$$\rho_0 \partial_t \boldsymbol{V}_1 = -\nabla P_1 + \mu_0^{-1} (\nabla \wedge \boldsymbol{B}_1) \wedge \boldsymbol{B}_0 \tag{26b}$$
$$-\partial_t \boldsymbol{B}_1 + \nabla \wedge (\boldsymbol{V}_1 \wedge \boldsymbol{B}_0) = 0, \tag{26c}$$

and the last equation gives $P_1/P_0 = \gamma \rho_1/\rho_0$. With Fourier modes, this yields:

$$-i\omega \rho_1 + i\rho_0 \boldsymbol{k} \cdot \boldsymbol{V}_1 = 0 \tag{27a}$$
$$-i\rho_0 \omega \boldsymbol{V}_1 = -i\boldsymbol{k} P_1 + \mu_0^{-1} (i\boldsymbol{k} \wedge \boldsymbol{B}_1) \wedge \boldsymbol{B}_0 \tag{27b}$$
$$i\omega \boldsymbol{B}_1 + i\boldsymbol{k} \wedge (\boldsymbol{V}_1 \wedge \boldsymbol{B}_0) = 0. \tag{27c}$$

We can thus express:

$$\rho_1 = \rho_0 \boldsymbol{k} \cdot \boldsymbol{V}_1/\omega \tag{28a}$$
$$\omega \boldsymbol{B}_1 = -\boldsymbol{k} \wedge (\boldsymbol{V}_1 \wedge \boldsymbol{B}_0) = (\boldsymbol{k} \cdot \boldsymbol{V}_1)\boldsymbol{B}_0 - (\boldsymbol{k} \cdot \boldsymbol{B}_0)\boldsymbol{V}_1, \tag{28b}$$
$$P_1 = \gamma P_0 \boldsymbol{k} \cdot \boldsymbol{V}_1/\omega. \tag{28c}$$

These three expressions can then be inserted into the equation of motion (Eq. 27b). It leads to a vector equation involving $\boldsymbol{V}_1$ only, as well as equilibrium quantities and the wavevector. Assuming that $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$ and that $\boldsymbol{k}$ lies in the x-z plane, and denoting the angle between them by θ, this equation can be written in matrix form:

$$\begin{pmatrix} \omega^2 - k^2v_A^2 - k^2v_s^2\sin^2\theta & 0 & -k^2v_s^2\sin\theta\cos\theta \\ 0 & \omega^2 - k^2v_A^2\cos^2\theta & 0 \\ -k^2V_s^2\sin\theta\cos\theta & 0 & \omega^2 - k^2v_s^2\cos^2\theta \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = 0. \quad (29)$$

We defined the Alfvén velocity $v_A = B_0/\sqrt{\mu_0\rho_0}$, and the sound speed $v_s = \sqrt{\gamma P_0/\rho_0}$. We also define $c_{ms}^2 = v_A^2 + v_s^2$. Let us denote the above matrix by M. There are non zero solutions only if $\det M = 0$ for some value of ω, and this gives the dispersion relation. We thus have:

$$(\omega^2 - k^2v_A^2\cos^2\theta)(\omega^4 - \omega^2k^2c_{ms}^2 + k^4v_A^2v_s^2\cos^2\theta) = 0. \quad (30)$$

The first root, $\omega = kv_A\cos\theta$, is the dispersion relation for **Alfvén waves**. When ω satisfy this relation, the vector for which $M\boldsymbol{V} = 0$ is $\boldsymbol{V} = (0, V_y, 0)$. It implies that this wave is transverse ($\boldsymbol{V}_1 \cdot \boldsymbol{k} = 0$), that it is not compressible ($\rho_1 = P_1 = 0$), and that $\boldsymbol{B}_1 \propto \boldsymbol{V}_1 \propto \hat{\boldsymbol{y}}$.

The second root is given by $\omega = kv_+$, with

$$v_+ = \left\{ \frac{1}{2}\left[c_{ms}^2 + \sqrt{c_{ms}^4 - 4v_A^2v_s^2\cos^2\theta} \right] \right\}^{1/2}. \quad (31)$$

This is the **fast magnetosonic wave** dispersion relation. When ω take this value, the vector such that $M\boldsymbol{V} = 0$ is of the form $\boldsymbol{V} = (V_x, 0, V_z)$ with the two components linked by the ratio $-M_{31}/M_{11}$ (or equally $-M_{33}/M_{31}$. It thus implies a finite P_1 and ρ_1, as well as an arbitrary polarization.

The last root is given by $\omega = kv_-$, with v_- similar to v_+ but with a minus in front of the $\sqrt{\cdot}$ of equation 31. This is the **slow magnetosonic wave**. The associated vector is again of the form $\boldsymbol{V} = (V_x, 0, V_z)$, with the same consequences.

2.5 Hot and magnetized plasma at arbitrary frequency

This case requires the use of kinetic theory and is much more complex. Landau damping is present as in the unmagnetized and warm case, for the absorption of waves with $\boldsymbol{E}_1 \propto \boldsymbol{B}_0$. In addition, waves with a circular polarization are also damped when their frequency is close to harmonics of the cyclotron frequencies: $\omega - \omega_{ce} \lesssim k_z v_{th,e}$ for the right-handed waves, and for $\omega - \omega_{ci} \lesssim k_z v_{th,i}$ for the left-handed waves.

The physics of this cyclotron damping is similar to that of the longitudinal waves Landau damping (but with differences, see Chen). Here, the resonance occurs when the wave polarization rotates close to the harmonics of the cyclotron pulsation of the particles of species s: on average, particles gyrating faster than the waves are slowed down and give energy to the wave, particles gyrating slower than the waves are accelerated and take energy to the wave. For a Maxwellian plasma, there are more particles going slowly, so that the wave is damped.

The appearance of all these resonnances allows for energy transfert from particles to waves at much more frequencies than in the unmagnetized case. In addition, cyclotron damping is more robust than unmagnetized Landau damping, and can happen also if the slope of the particle distribution function is not negative (Chen). This is important for example in fusion devices, where the plasma is heated by launching a given kind of waves in it.

Parallel propagation

- The electrostatic wave of the hot and unmagnetized case remains the same for parallel propagation, essentially because it implies oscillations parallel to $\boldsymbol{B}_0$, undisturbed by $\boldsymbol{B}_0$. It is identically Landau-damped whenever $\omega/k \lesssim v_{th,e}$,.

- The right and left-handed modes of the cold and magnetized case are again present (the blue mode and red mode of figure 2), with roughly the same properties. The main difference is that these modes are now cyclotron-damped when approching the cyclotron resonnances, i.e., for $\omega - \omega_{ce} \lesssim k_z v_{th,e}$ for the right-handed wave, and for $\omega - \omega_{ci} \lesssim k_z v_{th,i}$ for the left-handed wave.

  We remark that these absorptions were also found with the cold-magnetized plasma model, but with a zero width (they occured exactly at $\omega = \omega_{ce}$ and $\omega = \omega_{ci}$), while here the absorption width is finite.

Perpendicular propagation

- The electromagnetic or O-mode of cold plasma theory is still present, but modified by the appearance of cyclotron resonnances at $\omega = n\omega_{cs}$ (with $n \in \mathbb{Z}^*$). The wave is cyclotron-damped at these frequencies[11].

  The situation is however not so simple, because the weight (or the efficiency) of these resonnances depends on the temperature of the plasma[12], and, more specifically, on the ratio of the thermal Larmor radius of the particles to the wavelength, $k_\perp r_{cs}$. This is because the resonnances are a finite Larmor effect (FLR), i.e., due to the variation of the wave phase across a particle gyration. The weight of the nth resonnance in the dispersion relation is $(k_\perp r_{cs})^{|n|}$. If $k_\perp r_{cs} \to 0$, we are in the cold limit, and the resonnances disappear. For small but finite $k_\perp r_{cs}$, only the low-order resonnances persist. At $k_\perp r_{cs} \sim 1$, all resonnances contribute equally.

  Since $r_{ci} > r_{ce}$, ion resonnances are more important than electron resonnances.

- The X-mode is still present, but as we explain here, affected by cyclotron resonnances (especially for large wavenumbers for which $k_\perp r_{cs}$ is not small) and coupled to other modes. The other modes in question are called Bernstein waves, and appear only for $T \neq 0$. They can be derived in the electrostatic approximation, valid for small phase speeds $\omega/k_\perp \ll c$. There are electron Bernstein waves (for which the ion physics has little importance), and ion Bernstein waves (for which the electron physics play a role). Each mode has several branches, each asymptoting $n\omega_{cs}$ at large k. At large phase speed $\omega/k_\perp \sim c$ (small $k_\perp$), electromagnetic effects become important and these modes are no longer purely electrostatic. This is where they couple with the X-mode of the cold case.

  The full analysis leads to a single mode, shown in figure 1.

There are also more general oblique cyclotronic modes.

We do not elaborate more on this topic. Some elements are discussed in Chen or in Fitzpatrick (2011). The whole theory is contained in Stix (1992).

2.6 Formulary

We give the expressions for the pulsations and velocities appearing in Fig. 2.

[11] For strictly perpendicular propagation this is not strictly a cyclotron damping, but a conversion into a non-propagating oscillation, as was the case for the resonnance and absorption derived in the fluid model at ω_{uh} and ω_{lh}. For not strictly perpendicular propagation, cyclotron damping does happen.

[12] This is excpected, because the plasma temperature is a new parameter compared to the cold case, and the dispersion relations should obviously depend on it

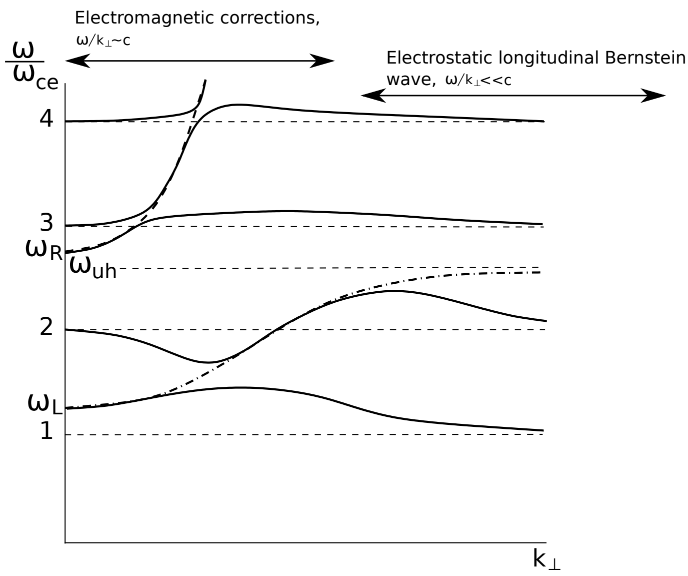

Figure 1: Full line: Dispersion relation of electron Bernstein waves, when the coupling with the X-mode at small k is taken into account. From Fitzpatrick (2011). The relative position of ω_L, ω_{uh}, ω_R with respect to the cyclotron harmonics is not always as such, and depends on ω_{ce}/ω_{pe}. Dashed line: X-mode of the cold plasma.

Pulsations of the cold wave theory

The pulsations appearing in the theory of cold magnetized plasmas are all a mix between the electron and ion plasma pulsations (collective plasma effects) and the electron and ion cyclotron pulsations (individual particle dynamics).

Lower hybrid pulsation:

$$\omega_{lh}^2 = \frac{\omega_{ce}^2\omega_{pi}^2 + \omega_{ce}^2\omega_{ci}^2 + \omega_{pe}^2\omega_{ci}^2}{\omega_{pe}^2 + \omega_{ce}^2}. \tag{32}$$

Upper hybrid pulsation:

$$\omega_{uh} = \left(\omega_{pe}^2 + \omega_{ce}^2\right)^{1/2}. \tag{33}$$

Right and left cutoff pulsations (where all pulsations are taken positive):

$$\begin{aligned} \omega_L &= \frac{1}{2}[(\omega_{ce}^2 + 4\omega_{pe}^2)^{1/2} - \omega_{ce}], \\ \omega_R &= \frac{1}{2}[(\omega_{ce}^2 + 4\omega_{pe}^2)^{1/2} + \omega_{ce}]. \end{aligned} \tag{34}$$

We have the ordering $\omega_R > \omega_{uh} > \omega_{pe} > \omega_L$, and also $\omega_L > \omega_{uh}$ for an ion-electron plasma below $\omega_{ce}/\omega_{pe} = 30$. See figure 3 for details.

Velocities of MHD waves

Within the MHD description, all waves have a linear dispersion relation, so that it is enough to just give the phase velocity w/k.

The non-relativistic Alfvén velocity is given by

$$v_A = \frac{B}{\sqrt{4\pi n_e m_i}}. \tag{35}$$

The magnetosonic velocity appearing in the MHD part of figure 2 (lower-left part) is defined as:

$$c_{\mathrm{ms}} = \left(v_A^2 + c_s^2\right)^{1/2}, \tag{36}$$

where v_A is defined by equation 35, and c_s is the speed of the ion sound wave, given by

$$c_s = \left(\frac{T_e + 3ZT_i}{m_i}\right)^{1/2} = \left(\frac{1}{2}\frac{m_e}{m_i}v_{th,e}^2 + \frac{3}{2}Zv_{th,i}^2\right)^{1/2}. \tag{37}$$

References

P. M. Bellan. Fundamentals of Plasma Physics. January 2006. URL http://adsabs.harvard.edu/abs/2006fpp..book.....B.

F. F. Chen. Introduction to plasma physics and controlled fusion. Vol. I: plasma physics. Second edition.

R. Fitzpatrick. The Physics of Plasmas. 2011. URL http://farside.ph.utexas.edu/teaching/plasma/plasma.html.

E. M. Lifshitz and L. P. Pitaevskii. Physical kinetics. 1981. URL http://adsabs.harvard.edu/abs/1981phki.book.....L.

K. T. McDonald. An electrostatic Wave. 2002. URL http://puhep1.princeton.edu/~mcdonald/examples/bernstein.pdf.

T. H. Stix. Waves in plasmas. 1992. URL http://adsabs.harvard.edu/abs/1992wapl.book.....S.

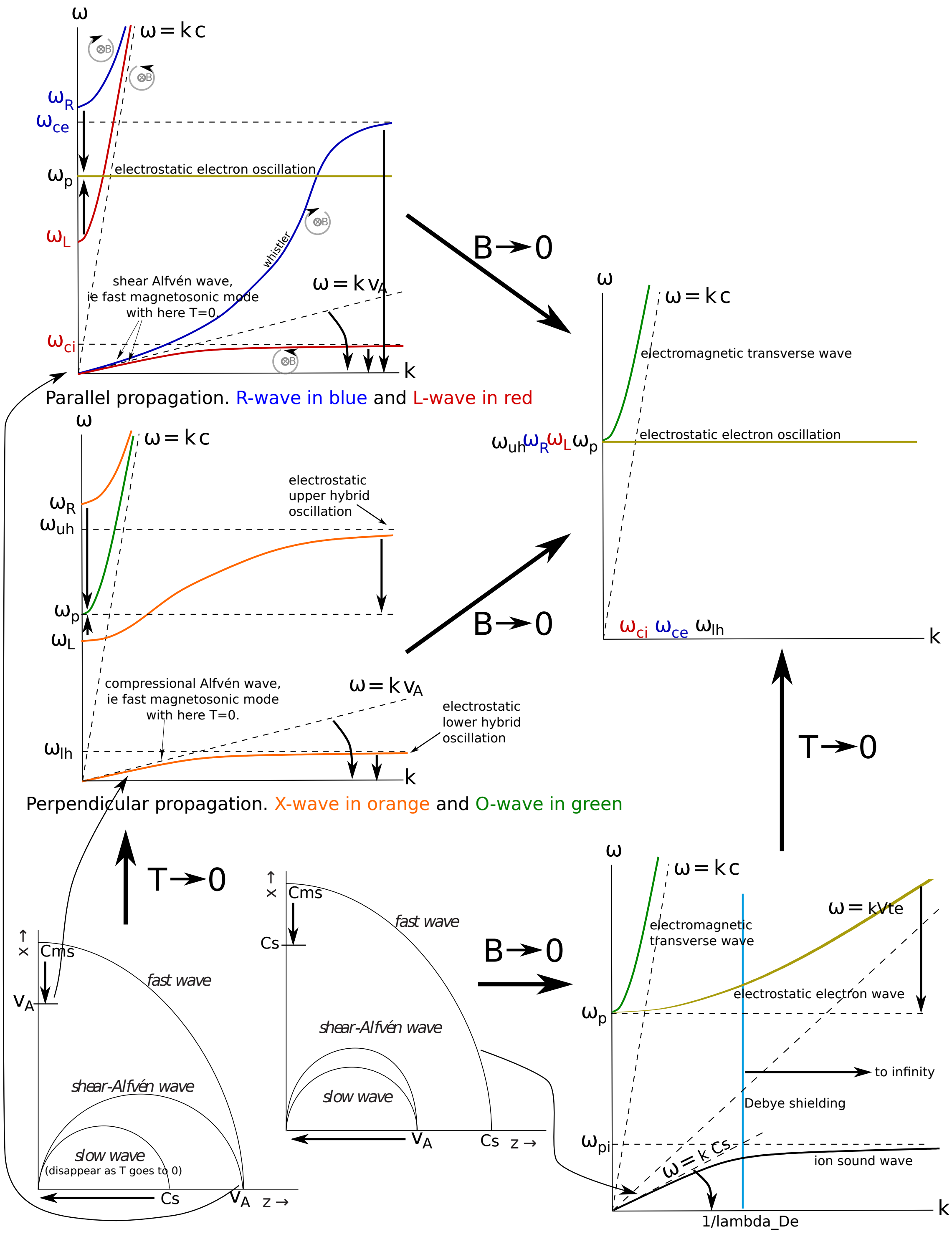

Figure 2: Different modes propagating in a homogeneous plasma. The black arrows show how the modes evolve when the temperature T or the background magnetic field B_0 goes to zero. For example, to go from the lower-right to the upper-right diagram, one has to make $T \to 0$. The modes of the lower-right diagram then evolve according to the black arrows of this diagram, which act to transform it into the upper-right diagram. In the MHD part, the magnetic field is directed along z.

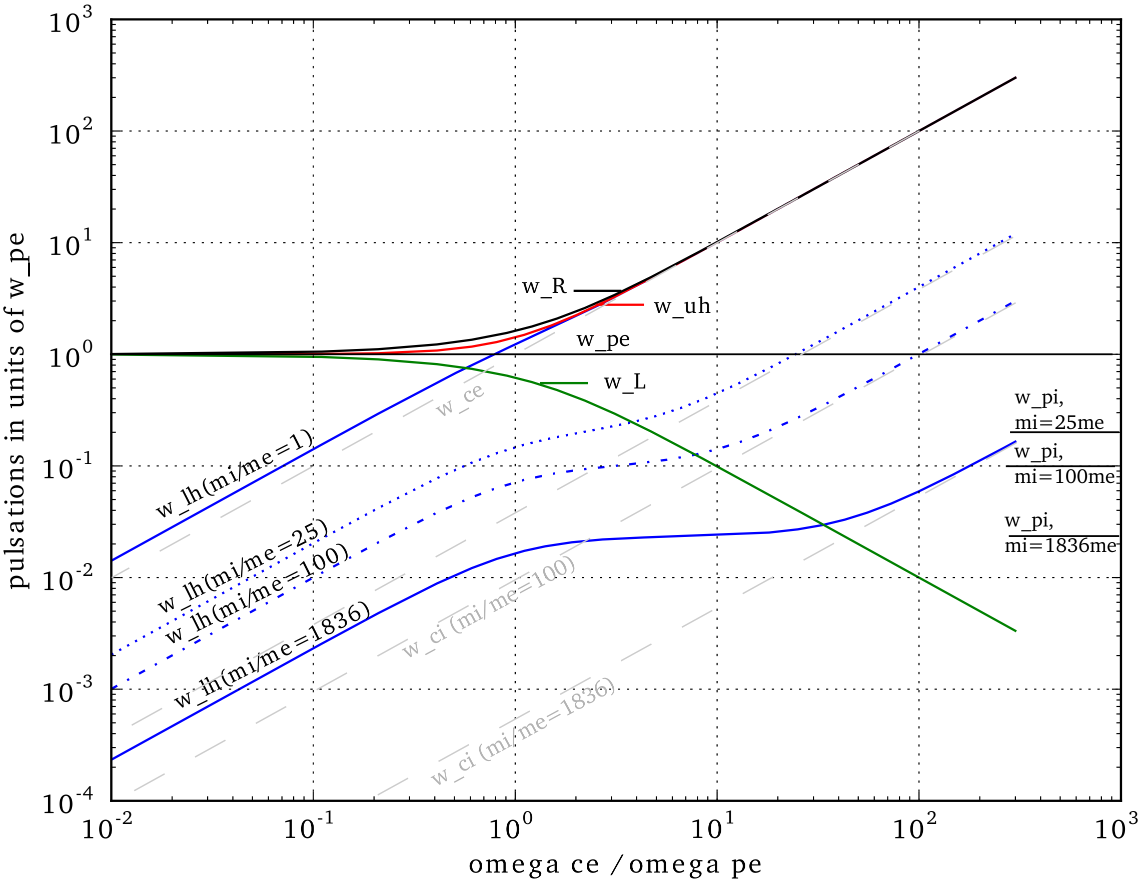

Figure 3: Pulsations of the cold wave therory.